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ELASTODYNAMIC ANALYSIS OF INHOMOGENEOUS ANISOTROPIC BODIES

Y. NIWA and S. HIROSE
Department of Civil Engineering, Kyoto University, Kyoto 606, Japan

and

M. KITAHARA
Department of Ocean Civil Engineering, Tokai University, Shimizu, Shizuoka 424, Japan

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Abstract—The integral equation method is presented for elastodynamic problems of inhomogeneous anisotropic bodies. Since fundamental solutions are not available for general inhomogeneous anisotropic media, we employ the fundamental solution for homogeneous elastostatics. The terms induced by material inhomogeneity and inertia force are regarded as body forces in elastostatics, and evaluated in the form of volume integrals. The scattering problems of elastic waves by inhomogeneous anisotropic inclusions are investigated for some test cases. Numerical results show the significant effects of inhomogeneity and anisotropy of materials on wave propagations.

1. INTRODUCTION

Over the last decade a number of developments in the boundary integral equation (BIE) method have been published in the field of elastodynamics. However, most of them have been devoted to homogeneous, or piecewise homogeneous isotropic bodies except for some special cases. For the antiplane problem of an *ad hoc* inhomogeneous medium, Niwa and Hirose[1] found the fundamental solution in an explicit form and applied it to the formulation of the BIE. In elastostatics, the direct BIE formulations were derived by using the explicit fundamental solutions to the governing field equations of anisotropic homogeneous media[2–4]. Moreover, Nishimura and Kobayashi[5] proposed the indirect BIE for three-dimensional anisotropic homogeneous bodies, which did not use an explicit expression of the fundamental solution. However, the applicability of the BIE method in elastodynamics has been mainly restricted to homogeneous isotropic bodies, since the fundamental solutions are not available for general inhomogeneous anisotropic media.

In general, governing equations for inhomogeneous anisotropic bodies become differential equations with variable coefficients. Some of the earliest applications of the integral equation method to solving differential equations with variable coefficients were done by Lautenbacher[6] and Mattioli[7]. They investigated the linearized long water waves with variable depth by using the fundamental solutions for the Helmholtz equation[6] and the Laplace equation[7]. Niwa *et al.*[8] applied the formulation proposed by Mattioli to antiplane problems of inhomogeneous elastic bodies, and furthermore extended it to inplane problems of inhomogeneous isotropic bodies[9–11]. In a series of these approaches, the fundamental solutions in homogeneous elastostatics were employed. Therefore, the terms with respect to material inhomogeneity and inertia force were regarded as body forces in elastostatics, and evaluated in the forms of volume integrals. Similar approaches were found in the analysis, e.g. of orthotropic plates[12] and of inelastic problems[13].

In the present paper, we further extend the above-mentioned approach to elastodynamic problems of general inhomogeneous anisotropic bodies. As numerical examples, in addition, we investigate the scattering problems of elastic waves by various inhomogeneous anisotropic inclusions in an infinite space.

2. FUNDAMENTAL EQUATIONS IN ELASTODYNAMICS

2.1. *Three-dimensional problem*

Let D denote an internal domain in the three-dimensional Euclidian space R^3 and ∂D denote its bounding surface. Under the assumption of infinitesimal deformations and linear elastic behavior of material, the equation of motion for an inhomogeneous anisotropic body D is

$$(C_{ijkl}(\mathbf{x})\hat{u}_{k,l}(\mathbf{x},t))_{,j}+\rho(\mathbf{x})\hat{b}_i(\mathbf{x},t)=\rho(\mathbf{x})\frac{\partial^2\hat{u}_i}{\partial t^2}(\mathbf{x},t)\quad \mathbf{x}\text{ in }D \tag{1}$$

where $\hat{u}_i$, $\hat{b}_i$, C_{ijkl}, ρ are displacement, body force, elastic moduli, and mass density, respectively. From the assumption of material inhomogeneity, elastic moduli C_{ijkl} and mass density ρ are inhomogeneous, i.e. vary from point to point in the body D. For general anisotropic materials, the number of independent constants of C_{ijkl} is 21. Isotropic materials are characterized by only two independent constants such as

$$C_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \tag{2}$$

where λ and μ are Lamé constants.

Here we consider the case in which all field quantities are harmonic in time with the angular frequency ω, i.e.

$$\begin{aligned}\hat{u}_i(\mathbf{x},t)&=Re\,\{u_i(\mathbf{x})\,e^{-i\omega t}\}\\ \hat{b}_i(\mathbf{x},t)&=Re\,\{b_i(\mathbf{x})\,e^{-i\omega t}\}.\end{aligned} \tag{3}$$

If we substitute eqn (3) into eqn (1), we obtain the governing equation in a steady state

$$(C_{ijkl}(\mathbf{x})u_{k,l}(\mathbf{x}))_{,j}+\rho(\mathbf{x})b_i(\mathbf{x})+\rho(\mathbf{x})\omega^2u_i(\mathbf{x})=0\quad \mathbf{x}\text{ in }D \tag{4}$$

where the time factor $\exp(-i\omega t)$ is suppressed.

For the mixed boundary value problem in a steady state, the boundary conditions are given as follows:

$$u_i(\mathbf{x})=\bar{u}_i(\mathbf{x})\quad \mathbf{x}\text{ on }\partial D_u \tag{5a}$$

$$\begin{aligned}t_i(\mathbf{x})&\equiv n_j(\mathbf{x})C_{ijkl}(\mathbf{x})u_{k,l}(\mathbf{x})\\ &=\bar{t}_i(\mathbf{x})\quad \mathbf{x}\text{ on }\partial D_t\end{aligned} \tag{5b}$$

where n_j is a unit vector normal to a boundary and $\partial D=\partial D_u+\partial D_t$. $\bar{u}_i$ and $\bar{t}_i$ are prescribed quantities.

2.2. *Two-dimensional problem*

If all field quantities in question depend only on two space variables, say x_1 and x_2, then the problem is reduced to a two-dimensional problem in a plane strain. In this case, the equation of motion takes the following form

$$(C_{i\beta k\gamma}(\mathbf{x})u_{k,\gamma}(\mathbf{x}))_{,\beta}+\rho(\mathbf{x})b_i(\mathbf{x})+\rho(\mathbf{x})\omega^2u_i(\mathbf{x})=0\quad \mathbf{x}\text{ in }D \tag{6}$$

where Roman indices are used for 1, 2 and 3 and Greek indices for 1 and 2 only. Furthermore, the boundary conditions are expressed as

$$u_i(\mathbf{x})=\bar{u}_i(\mathbf{x})\quad \mathbf{x}\text{ on }\partial D_u \tag{7a}$$

$$\begin{aligned}t_i(\mathbf{x})&=n_\beta(\mathbf{x})C_{i\beta k\gamma}(\mathbf{x})u_{k,\gamma}(\mathbf{x})\\ &=\bar{t}_i(\mathbf{x})\quad \mathbf{x}\text{ on }\partial D_t.\end{aligned} \tag{7b}$$

From eqns (6) and (7), the inplane motions u_α and the antiplane motion u_3 are coupled for the two-dimensional problem considered here.

3. POTENTIALS

Nishimura and Kobayashi[14] investigated elastic potentials elaborately in order to solve elastoplastic problems. In this section, we follow them to show some important properties of elastic potentials which will be used later on. For simplicity, the surface ∂D is assumed to be a sufficiently smooth surface in R^N ($N = 2$ or 3).

3.1. *Fundamental solutions*

In general, it is difficult to find fundamental solutions for inhomogeneous bodies. In our integral formulation, therefore, we employ the fundamental solutions for homogeneous bodies in elastostatics, which satisfy

$$\begin{aligned}\Delta^*_{ij}(\partial)U^m_j(\mathbf{x},\mathbf{y}) &\equiv C^*_{ikjl}U^m_{j,kl}(\mathbf{x},\mathbf{y}) \\ &= -\delta_{im}\delta(\mathbf{x}-\mathbf{y})\end{aligned} \tag{8}$$

where C^*_{ikjl} do not depend on the position and denote components of a tensor of rank 4 with the symmetries

$$C^*_{ikjl} = C^*_{kijl} = C^*_{iklj} = C^*_{jlik}.$$

Moreover, δ_{im} is the Kronecker delta and $\delta(\mathbf{x}-\mathbf{y})$ is the Dirac delta function. For the special case of isotropy, i.e.

$$C^*_{ikjl} = \lambda^*\delta_{ik}\delta_{jl}+\mu^*(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{kj}) \tag{9}$$

the fundamental solutions U^m_i are found in the closed forms

(three-dimensional problem)

$$U^m_i(\mathbf{x},\mathbf{y}) = \frac{1}{8\pi\mu^*(\lambda^*+2\mu^*)r}\{(\lambda^*+3\mu^*)\delta_{im}+(\lambda^*+\mu^*)r_{,i}r_{,m}\} \tag{10a}$$

(two-dimensional problem)

$$U^\sigma_\alpha(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi\mu^*(\lambda^*+2\mu^*)}\left\{(\lambda^*+3\mu^*)\delta_{\alpha\sigma}\ \ln\frac{1}{r}+(\lambda^*+\mu^*)r_{,\alpha}r_{,\sigma}\right\} \tag{10b}$$

$$U^3_3(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi\mu^*}\ln\frac{1}{r}$$

where $r = |\mathbf{x}-\mathbf{y}|$.

3.2. *Simple layer potential*

For a continuous density ϕ_m, the simple layer potential

$$V_i(\mathbf{x}) = \int_{\partial D} U^m_i(\mathbf{x},\mathbf{y})\phi_m(\mathbf{y})\,\mathrm{d}S_y \tag{11}$$

is continuous everywhere in a space R^N.

3.3. *Double layer potential*

Next, we consider the double layer potential defined as

$$W_i(\mathbf{x}) = \int_{\partial D} U^{m,j}_i(\mathbf{x},\mathbf{y})\psi_{mj}(\mathbf{y})\,\mathrm{d}S_y \tag{12}$$

where $\psi_{mj}(\mathbf{y})$ is a sufficiently well-behaving density. The superscript ^{,j} indicates the differential with respect to the second argument of U_i^m, i.e. $\partial/\partial y_j$. This potential shows the following finite limit W_i^+ (W_i^-) as $\mathbf{x}$ approaches a boundary point on ∂D from the side into which the positive (negative) normal points

$$W_i^{\pm}(\mathbf{x}) = \pm\tfrac{1}{2}n_j\Delta^{*-1}_{im}(\mathbf{n})\psi_{mj}(\mathbf{x}) + \fint_{\partial D} U_i^{m,j}(\mathbf{x},\mathbf{y})\psi_{mj}(\mathbf{y})\,\mathrm{d}S_y \quad \mathbf{x} \text{ on } \partial D \tag{13}$$

where $\Delta^{*-1}_{im}(\boldsymbol{\xi})$ denotes the inverse matrix of Fourier transform of Navier's operator $\Delta^*_{im}(\partial)$ defined in eqn (8), and $\fint \cdot \mathrm{d}s$ indicates the principal value integral. For the isotropic case as shown in eqn (9), $\Delta^{*-1}_{im}(\mathbf{n})$ has the explicit form

$$\Delta^{*-1}_{im}(\mathbf{n}) = \frac{1}{\mu^*}\left(\delta_{im} - \frac{\lambda^*+\mu^*}{\lambda^*+2\mu^*}n_in_m\right). \tag{14}$$

3.4. *Volume potential*

The volume potentials

$$D_i(\mathbf{x}) = \int_D U_i^m(\mathbf{x},\mathbf{y})\phi_m(\mathbf{y})\,\mathrm{d}V_y \tag{15}$$

$$D_i(\mathbf{x}) = \int_D U_i^{m,j}(\mathbf{x},\mathbf{y})\psi_{mj}(\mathbf{y})\,\mathrm{d}V_y \tag{16}$$

are continuous everywhere in the space R^N.

The volume potential defined as

$$D_i''(\mathbf{x}) = \int_D U_i^{m,jk}(\mathbf{x},\mathbf{y})\varphi_{mjk}(\mathbf{y})\,\mathrm{d}V_y \tag{17}$$

can be written as

$$D_i''(\mathbf{x}) = \begin{cases} -v_{kimj}\varphi_{mjk}(\mathbf{x}) + \fint_D U_i^{m,jk}(\mathbf{x},\mathbf{y})\varphi_{mjk}(\mathbf{y})\,\mathrm{d}V_y & \mathbf{x} \text{ in } D \quad (18\text{a}) \\ \int_D U_i^{m,jk}(\mathbf{x},\mathbf{y})\varphi_{mjk}(\mathbf{y})\,\mathrm{d}V_y & \mathbf{x} \text{ in } D^C \quad (18\text{b}) \end{cases}$$

(D^C: complement of D)

where v_{ijkl} is a tensor defined as

$$v_{ijkl} = \frac{1}{|S^N|}\int_{S^N} \xi_i\Delta^{*-1}_{jk}(\boldsymbol{\xi})\xi_l\,\mathrm{d}S_\xi \tag{19}$$

(S^N: N-dimensional unit sphere; $|S^N|$: area of S^N).

The term $-v_{kimj}\varphi_{mjk}(\mathbf{x})$ in eqn (18a) is the convected term from the singular integral[15]. For the special case of isotropy, v_{ijkl} is expressed in the following form

$$v_{ijkl} = \frac{\{N(\lambda^*+2\mu^*)+(\lambda^*+3\mu^*)\}\delta_{il}\delta_{jk} - (\lambda^*+\mu^*)(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl})}{\mu^*(\lambda^*+2\mu^*)N(N+2)}. \tag{20}$$

Furthermore, $D_i''(\mathbf{x})$ shows the following discontinuity across the boundary ∂D

$$D_i''^{\pm}(\mathbf{x}) = -\tfrac{1}{2}v_{kimj}\varphi_{mjk}(\mathbf{x}) \pm \tfrac{1}{2}n_k\Delta_{im}^{*-1}(\mathbf{n})n_j\varphi_{mjk}(\mathbf{x}) + \int_D U_i^{m,jk}(\mathbf{x},\mathbf{y})\varphi_{mjk}(\mathbf{y})\,\mathrm{d}V_y \quad \mathbf{x}\text{ on }\partial D. \tag{21}$$

4. FORMULATION OF INTEGRAL EQUATIONS FOR INHOMOGENEOUS ANISOTROPIC BODIES

In this section, integral representations of displacements in an inhomogeneous anisotropic body are derived on the basis of a static fundamental solution for a homogeneous body. For simplicity, the elastic moduli C_{ijkl} are assumed to be continuous everywhere in the domain D.

4.1. *Threc-dimensional problem*

With use of the constants C^*_{ijkl} defined in eqn (8), the governing eqn (4) is written in the form,

$$C^*_{ijkl}\,u_{k,lj}(\mathbf{x}) = -F_i(\mathbf{x}) \quad \mathbf{x}\text{ in }D$$

where $F_i(\mathbf{x}) = \rho(\mathbf{x})b_i(\mathbf{x}) + \rho(\mathbf{x})\omega^2u_i(\mathbf{x}) + \{(C_{ijkl}(\mathbf{x}) - C^*_{ijkl})u_{k,l}(\mathbf{x})\}_{,j}$. The term F_i is considered as a body force in elastostatics.

We now multiply the above equation by the fundamental solution U_i^m which satisfies eqn (8) for a homogeneous anisotropic medium, and integrate all over the domain D. Then we obtain

$$\int_D U_i^m(\mathbf{x},\mathbf{y})\,\{C^*_{ijkl}u_{k,lj}(\mathbf{x}) + F_i(\mathbf{x})\}\,\mathrm{d}V_x = 0 \tag{22}$$

or in a slightly different form

$$\int_D \{(U_i^m(\mathbf{x},\mathbf{y})\,C_{ijkl}(\mathbf{x})u_{k,l}(\mathbf{x}))_{,j} - (U_{i,j}^m(\mathbf{x},\mathbf{y})C_{ijkl}(\mathbf{x})u_k(\mathbf{x}))_{,l}\}\,\mathrm{d}V_x$$
$$+\int_D \{(U_{i,j}^m(\mathbf{x},\mathbf{y})C_{ijkl}(\mathbf{x}))_{,l}u_k(\mathbf{x})\}\,\mathrm{d}V_x$$
$$+\int_D U_i^m(\mathbf{x},\mathbf{y})\,(\rho(\mathbf{x})b_i(\mathbf{x}) + \rho(\mathbf{x})\omega^2u_i(\mathbf{x}))\,\mathrm{d}V_x = 0. \tag{23}$$

Applying the divergence theorem to the first volume integral of eqn (23) and using the relation $U_i^m(\mathbf{x},\mathbf{y}) = U_m^i(\mathbf{y},\mathbf{x})$, we have

$$\int_{\partial D} \{U_m^i(\mathbf{x},\mathbf{y})t_i(\mathbf{y}) - T_m^i(\mathbf{x},\mathbf{y})u_i(\mathbf{y})\}\,\mathrm{d}S_y + \int_D (U_m^{i,j}(\mathbf{x},\mathbf{y})C_{ijkl}(\mathbf{y}))^{,l}u_k(\mathbf{y})\,\mathrm{d}V_y$$
$$+\int_D U_m^i(\mathbf{x},\mathbf{y})\,(\rho(\mathbf{y})b_i(\mathbf{y}) + \rho(\mathbf{y})\omega^2u_i(\mathbf{y}))\,\mathrm{d}V_y = 0 \tag{24}$$

where $T_m^i(\mathbf{x},\mathbf{y})$ is defined as

$$T_m^i(\mathbf{x},\mathbf{y}) = n_l(\mathbf{y})C_{kjil}(\mathbf{y})U_m^{k,j}(\mathbf{x},\mathbf{y}). \tag{25}$$

With the use of properties of potentials shown in the previous section, eqn (24) is rewritten as follows:

$$\int_{\partial D}\{U^i_m(\mathbf{x},\mathbf{y})t_i(\mathbf{y})-T^i_m(\mathbf{x},\mathbf{y})u_i(\mathbf{y})\}\,\mathrm{d}S_y$$

$$+\oint_D U^{i,jl}_m(\mathbf{x},\mathbf{y})C_{ijkl}(\mathbf{y})u_k(\mathbf{y})\,\mathrm{d}V_y+\int_D U^{i,j}_m(\mathbf{x},\mathbf{y})C_{ijkl,l}(\mathbf{y})u_k(\mathbf{y})\,\mathrm{d}V_y$$

$$+\int_D U^i_m(\mathbf{x},\mathbf{y})\rho(\mathbf{y})b_i(\mathbf{y})\,\mathrm{d}V_y+\omega^2\int_D U^i_m(\mathbf{x},\mathbf{y})\rho(\mathbf{y})u_i(\mathbf{y})\,\mathrm{d}V_y$$

$$=v_{lmij}C_{ijkl}(\mathbf{x})u_k(\mathbf{x})\quad \mathbf{x}\text{ in }D;\quad (26a)$$

$$\int_{\partial D}U^i_m(\mathbf{x},\mathbf{y})t_i(\mathbf{y})\,\mathrm{d}S_y-\oint_{\partial D}T^i_m(\mathbf{x},\mathbf{y})u_i(\mathbf{y})\,\mathrm{d}S_y$$

$$+\oint_D U^{i,jl}_m(\mathbf{x},\mathbf{y})C_{ijkl}(\mathbf{y})u_k(\mathbf{y})\,\mathrm{d}V_y+\int_D U^{i,j}_m(\mathbf{x},\mathbf{y})C_{ijkl,l}(\mathbf{y})u_k(\mathbf{y})\,\mathrm{d}V_y$$

$$+\int_D U^i_m(\mathbf{x},\mathbf{y})\rho(\mathbf{y})b_i(\mathbf{y})\mathrm{d}V_y+\omega^2\int_D U^i_m(\mathbf{x},\mathbf{y})\rho(\mathbf{y})u_i(\mathbf{y})\,\mathrm{d}V_y$$

$$=\tfrac{1}{2}v_{lmij}C_{ijkl}(\mathbf{x})u_k(\mathbf{x})\quad \mathbf{x}\text{ on }\partial D;\quad (26b)$$

$$\int_{\partial D}\{U^i_m(\mathbf{x},\mathbf{y})t_i(\mathbf{y})-T^i_m(\mathbf{x},\mathbf{y})u_i(\mathbf{y})\}\,\mathrm{d}S_y$$

$$+\int_D (U^{i,j}_m(\mathbf{x},\mathbf{y})C_{ijkl}(\mathbf{y}))^{,l}u_k(\mathbf{y})\,\mathrm{d}V_y+\int_D U^i_m(\mathbf{x},\mathbf{y})\rho(\mathbf{y})b_i(\mathbf{y})\,\mathrm{d}V_y$$

$$+\omega^2\int_D U^i_m(\mathbf{x},\mathbf{y})\rho(\mathbf{y})u_i(\mathbf{y})\,\mathrm{d}V_y=0\quad \mathbf{x}\text{ in }D^C;\quad (26c)$$

where v_{lmij} is defined in eqn (19).

Equations (26a) and (26b) constitute the boundary-domain integral equations with u_i in D and u_i and t_i on ∂D as unknowns.

4.2. *Two-dimensional problem*

In this section, the fundamental solutions for the isotropic bodies defined in eqns (10b) and (10c) are used in order to formulate integral equations for two-dimensional inhomogeneous anisotropic media with no loss of generality.

First of all, the governing eqn (6) for the two-dimensional problem can be written as follows

$$C^*_{\alpha\beta k\gamma}u_{k,\gamma\beta}(\mathbf{x})=-F_\alpha(\mathbf{x})\quad \mathbf{x}\text{ in }D \quad (27a)$$

$$C^*_{3\beta k\gamma}u_{k,\gamma\beta}(\mathbf{x})=-F_3(\mathbf{x})\quad \mathbf{x}\text{ in }D \quad (27b)$$

where

$$F_\alpha(\mathbf{x})=\rho(\mathbf{x})b_\alpha(\mathbf{x})+\rho(\mathbf{x})\omega^2u_\alpha(\mathbf{x})+\{(C_{\alpha\beta k\gamma}(\mathbf{x})-C^*_{\alpha\beta k\gamma})u_{k,\gamma}(\mathbf{x})\}_{,\beta}$$

$$F_3(\mathbf{x})=\rho(\mathbf{x})b_3(\mathbf{x})+\rho(\mathbf{x})\omega^2u_3(\mathbf{x})+\{(C_{3\beta k\gamma}(\mathbf{x})-C^*_{3\beta k\gamma})u_{k,\gamma}(\mathbf{x})\}_{,\beta}.$$

Multiplying eqn (27a) by U^σ_α defined in eqn (10b), and eqn (27b) by U^3_3 defined in eqn (10c), and integrating them over the domain D, we have

$$\int_D U_\alpha^\sigma(\mathbf{x},\mathbf{y})\,\{C^*_{\alpha\beta k\gamma}u_{k,\gamma\beta}(\mathbf{x})+F_\alpha(\mathbf{x})\}\,\mathrm{d}S_x=0 \tag{28a}$$

$$\int_D U_3^3(\mathbf{x},\mathbf{y})\,\{C^*_{3\beta k\gamma}u_{k,\gamma\beta}(\mathbf{x})+F_3(\mathbf{x})\}\,\mathrm{d}S_x=0. \tag{28b}$$

We follow the same procedures as those from eqn (22) to eqn (26). Then we obtain the following integral equations

$$\int_{\partial D}\{U_\sigma^\alpha(\mathbf{x},\mathbf{y})t_\alpha(\mathbf{y})-T_\sigma^k(\mathbf{x},\mathbf{y})u_k(\mathbf{y})\}\,\mathrm{d}s_y+\int_D(U_\sigma^{\alpha,\beta}(\mathbf{x},\mathbf{y})C_{\alpha\beta k\gamma}(\mathbf{y}))^{,\gamma}u_k(\mathbf{y})\,\mathrm{d}S_y$$
$$+\int_D U_\sigma^\alpha(\mathbf{x},\mathbf{y})\rho(\mathbf{y})b_\alpha(\mathbf{y})\,\mathrm{d}S_y+\omega^2\int_D U_\sigma^\alpha(\mathbf{x},\mathbf{y})\rho(\mathbf{y})u_\alpha(\mathbf{y})\,\mathrm{d}S_y$$
$$=\begin{cases} v_{\gamma\sigma\alpha\beta}C_{\alpha\beta k\gamma}(\mathbf{x})u_k(\mathbf{x}) & \mathbf{x}\text{ in }D \quad (29a)\\ \frac{1}{2}v_{\gamma\sigma\alpha\beta}C_{\alpha\beta k\gamma}(\mathbf{x})u_k(\mathbf{x}) & \mathbf{x}\text{ on }\partial D \quad (29b)\\ 0 & \mathbf{x}\text{ in }D^C \quad (29c)\end{cases}$$

$$\int_{\partial D}\{U_3^3(\mathbf{x},\mathbf{y})t_3(\mathbf{y})-T_3^k(\mathbf{x},\mathbf{y})u_k(\mathbf{y})\}\,\mathrm{d}s_y+\int_D(U_3^{3,\beta}(\mathbf{x},\mathbf{y})C_{3\beta k\gamma}(\mathbf{y}))^{,\gamma}u_k(\mathbf{y})\,\mathrm{d}S_y$$
$$+\int_D U_3^3(\mathbf{x},\mathbf{y})\rho(\mathbf{y})b_3(\mathbf{y})\,\mathrm{d}S_y+\omega^2\int_D U_3^3(\mathbf{x},\mathbf{y})\rho(\mathbf{y})u_3(\mathbf{y})\,\mathrm{d}S_y$$
$$=\begin{cases} \dfrac{1}{2\mu^*}C_{3\beta k\beta}(\mathbf{x})u_k(\mathbf{x}) & \mathbf{x}\text{ in }D \quad (30a)\\ \dfrac{1}{4\mu^*}C_{3\beta k\beta}(\mathbf{x})u_k(\mathbf{x}) & \mathbf{x}\text{ on }\partial D \quad (30b)\\ 0 & \mathbf{x}\text{ in }D^C \quad (30c)\end{cases}$$

where T_σ^k and T_3^k are defined as

$$T_\sigma^k(\mathbf{x},\mathbf{y})=n_\gamma(\mathbf{y})C_{\alpha\beta k\gamma}(\mathbf{y})U_\sigma^{\alpha,\beta}(\mathbf{x},\mathbf{y}) \tag{31a}$$

$$T_3^k(\mathbf{x},\mathbf{y})=n_\gamma(\mathbf{y})C_{3\beta k\gamma}(\mathbf{y})U_3^{3,\beta}(\mathbf{x},\mathbf{y}) \tag{31b}$$

respectively, and $v_{\gamma\sigma\alpha\beta}$ is given in eqn (20) where $N=2$. Note that some integrals in eqns (29) and (30) stand for their principal values as in eqn (26). It is clear that eqns (29) and (30) constitute the simultaneous integral equations having u_i in D and u_i and t_i on ∂D as unknowns, so that the inplane motion and the antiplane motion are not independent for these general anisotropic bodies.

5. SCATTERING OF ELASTIC WAVES BY INHOMOGENEOUS ANISOTROPIC INCLUSIONS IN AN INFINITE SPACE

Here we give some numerical examples in order to demonstrate the applicability of the integral equations derived in the previous section.

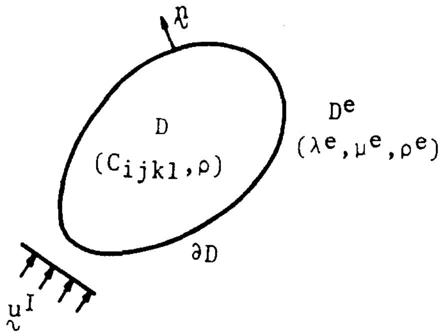

Fig. 1. An inhomogeneous anisotropic inclusion D in an infinite space D^e.

5.1. *Statement of the problem*

Let us consider the two-dimensional scattering problem of elastic waves by an inclusion D in an infinite space, as shown in Fig. 1. The inclusion D is assumed to be an arbitrary shaped inhomogeneous anisotropic body, while its surrounding domain D^e is assumed to be a homogeneous isotropic medium. The elastic constants and the density in the domain D^e are denoted by λ^e, μ^e, and ρ^e, and those of the inclusion by C_{ijkl} and ρ. The incident wave $\mathbf{u}^I$ is assumed to be a harmonic plane wave. Furthermore, we assume the continuities of displacements and tractions across the boundary ∂D.

5.2. *Integral formula for an exterior domain D^e*

Since the domain D^e is homogeneous and isotropic, the conventional Green's (Somigliana's) formula for an exterior domain leads to the following integral equations for the inplane motion and the antiplane motion

$$\int_{\partial D} \{H^{\alpha}_{\sigma}(\mathbf{x},\mathbf{y})u_{\alpha}(\mathbf{y}) - G^{\alpha}_{\sigma}(\mathbf{x},\mathbf{y})t_{\alpha}(\mathbf{y})\}\,\mathrm{d}s_y + u^I_{\alpha}(\mathbf{x}) = \begin{cases} u_{\alpha}(\mathbf{x}) & \mathbf{x} \text{ in } D^e \quad (32a) \\ \tfrac{1}{2}u_{\alpha}(\mathbf{x}) & \mathbf{x} \text{ on } \partial D \quad (32b) \\ 0 & \mathbf{x} \text{ in } (D^e)^C \quad (32c) \end{cases}$$

$$\int_{\partial D} \{H^{3}_{3}(\mathbf{x},\mathbf{y})u_{3}(\mathbf{x}) - G^{3}_{3}(\mathbf{x},\mathbf{y})t_{3}(\mathbf{y})\}\,\mathrm{d}s_y + u^I_{3}(\mathbf{x}) = \begin{cases} u_{3}(\mathbf{x}) & \mathbf{x} \text{ in } D^e \quad (33a) \\ \tfrac{1}{2}u_{3}(\mathbf{x}) & \mathbf{x} \text{ on } \partial D \quad (33b) \\ 0 & \mathbf{x} \text{ in } (D^e)^C \quad (33c) \end{cases}$$

where G^{α}_{σ} and G^3_3 are the fundamental solutions in elastodynamics, and H^{α}_{σ} and H^3_3 denote the kernels of the double layer potentials defined as

$$H^{\alpha}_{\sigma}(\mathbf{x},\mathbf{y}) = \lambda^e n_{\alpha}(\mathbf{y})G^{\beta,\beta}_{\sigma}(\mathbf{x},\mathbf{y}) + \mu^e n_{\beta}(\mathbf{y})\,(G^{\alpha,\beta}_{\sigma}(\mathbf{x},\mathbf{y}) + G^{\beta,\alpha}_{\sigma}(\mathbf{x},\mathbf{y})) \qquad (34a)$$

$$H^3_3(\mathbf{x},\mathbf{y}) = n_{\gamma}(\mathbf{y})\mu^e G^{3,\gamma}_3(\mathbf{x},\mathbf{y}). \qquad (34b)$$

The fundamental solutions G^{α}_{σ} and G^3_3 are well known to have the closed forms

$$G^{\alpha}_{\sigma}(\mathbf{x},\mathbf{y}) = \frac{i}{4\mu^e}\left[H^{(1)}_0(k^e_T r)\delta_{\sigma\alpha} + \frac{1}{(k^e_T)^2}\partial_{\sigma}\,\partial_{\alpha}\{H^{(1)}_0(k^e_T r) - H^{(1)}_0(k^e_L r)\}\right] \qquad (35a)$$

$$G^3_3(\mathbf{x},\mathbf{y}) = \frac{i}{4\mu^e}H^{(1)}_0(k^e_T r) \qquad (35b)$$

where $H^{(1)}_0(\cdot)$ denotes the zero order Hankel function of the first kind, and $k^e_T = \omega\,(\mu^e/\rho^e)^{-1/2}$ and $k^e_L = \omega((\lambda^e + 2\mu^e)/\rho)^{-1/2}$.

5.3. *Some remarks on the numerical implementation*

The boundary integral equations (32b) and (33b) are coupled with the boundary-domain integral equations (29) and (30) by taking account of the continuity conditions of displacements and tractions on the boundary ∂D. In the numerical analysis, these integral equations are discretized and reduced to an algebraic linear system with unknowns of displacements in D and displacements and tractions on ∂D. The procedure of discretization is quite similar to that shown for the case of inhomogeneous isotropic bodies[9, 10].

Some remarks on the numerical implementation are summarized here.

(1) The boundary ∂D is divided into N_B straight boundary elements and the domain D into N_D triangular body elements. These elements must be fine so as to represent the incident wave correctly. In the numerical examples presented later, $N_B = 48$ and $N_D = 270$.

(2) The quantities u_i and t_i are assumed to be constant over each element.

(3) We use the analytical method[14,16] to evaluate the integral over a singular element where the field point coincides with the source point. Except for the singular element, integrals over boundary elements are evaluated by the eight point Gaussian quadrature method, and integrals over body cells are evaluated by the seven point Gaussian quadrature method for the triangles.

(4) In order to avoid the cancellation which may occur in the dynamic fundamental solution defined in eqn (35a) when its argument is small, we expand it into a series and delete the apparent singularity before numerical integration.

(5) Since we employ the static fundamental solution in the formulation for inhomogeneous anisotropic inclusions, the frequency parameter is free from the integrals. Therefore, when dynamic responses are required for many frequencies, we evaluate the integrals once with respect to the static fundamental solution and store them for the repeated calculations.

5.4. *Numerical examples*

5.4.1. *Accuracy of solutions.* In the case of a homogeneous isotropic inclusion, the scattering problem can be also formulated by the conventional BIE method[17, 18]. To confirm the accuracy of solutions obtained by the present method, we can compare our results with the BIE solutions in the case of a homogeneous isotropic inclusion. In the following two examples, material constants are given as

$$\lambda^i/\mu^e = \mu^i/\mu^e = 0.5, \quad \rho^i/\rho^e = 1.0, \quad \nu^e = 0.25$$

where λ^i and μ^i are Lamé constants of the inclusion, and ν^e denotes the Poisson's ratio in the surrounding medium D^e.

Figure 2 shows the deformation of a circular cylindrical inclusion when the peak of a harmonic incident wave arrives at the center of the inclusion. The incident wave is assumed to be a plane P wave with the wave number $ak_T^e = \pi$ or 2π, where a is a radius of the inclusion. Figure 3 shows the deformation of a circular cylindrical inclusion subjected to a harmonic plane SV wave. In both figures, the results by the present method have a good agreement with those by the BIE method.

5.4.2. *Inhomogeneous inclusions.* To investigate the effect of material inhomogeneity on the scattering of elastic waves, three types of circular cylindrical inclusions are considered. One is a homogeneous isotropic inclusion, and the others are inhomogeneous isotropic inclusions. The material constants used are as follows:

(case 1)

$$\lambda^i/\mu^e = \mu^i/\mu^e = 0.5, \quad \rho^i/\rho^e = 1.0, \quad \nu^e = 0.25$$

(case 2)

$$\lambda^i/\mu^e = \mu^i/\mu^e = 0.5-0.4x_2/a, \quad \rho^i/\rho^e = 1.0, \quad \nu^e = 0.25$$

(case 3)

$$\lambda^i/\mu^e = \mu^i/\mu^e = 0.5+0.4x_2/a, \quad \rho^i/\rho^e = 1.0, \quad \nu^e = 0.25$$

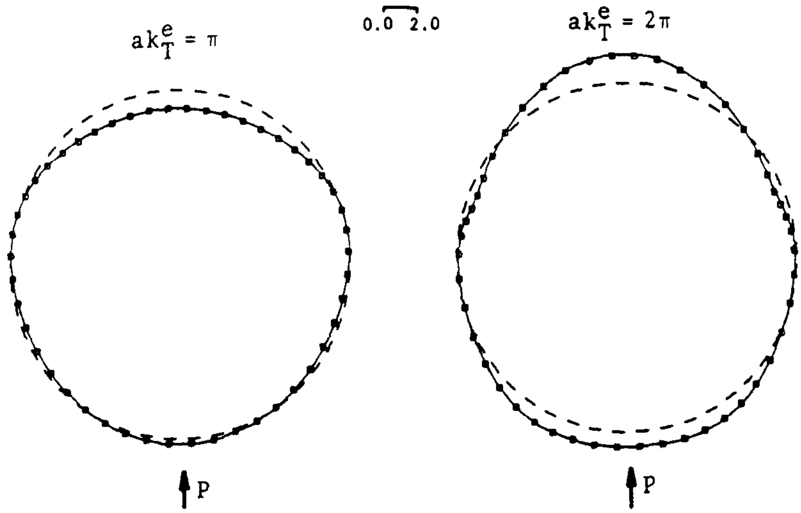

Fig. 2. Deformation of a circular cylindrical inclusion subjected to a plane P wave. The parameters used are $\lambda^i/\mu^e = \mu^i/\mu^e = 0.5$, $\rho^i/\rho^e = 1.0$, and $\nu^e = 0.25$. —: BIE method, □: present method.

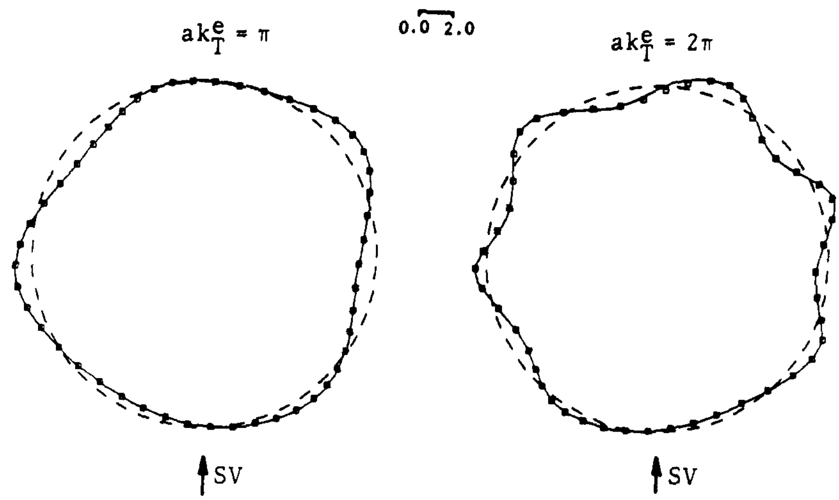

Fig. 3. Deformation of a circular cylindrical inclusion subjected to a plane SV wave. The parameters used are the same as those in Fig. 2. —: BIE method, □: present method.

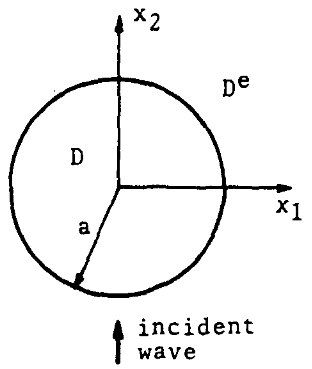

Fig. 4. A circular cylindrical inclusion.

where $x_1 - x_2$ coordinates are taken as shown in Fig. 4. The incident wave is assumed to be a sinusoidal plane SV wave with the wave number $ak_T^c = \pi$ and propagates to the x_2-direction.

The deformation of the inclusion corresponding to cases 1–3 is shown in Figs 5(a)–(c), respectively. On the right-hand side of these figures, the shape of the travelling incident wave is depicted at each time. The time t is normalized by the period T of the incident

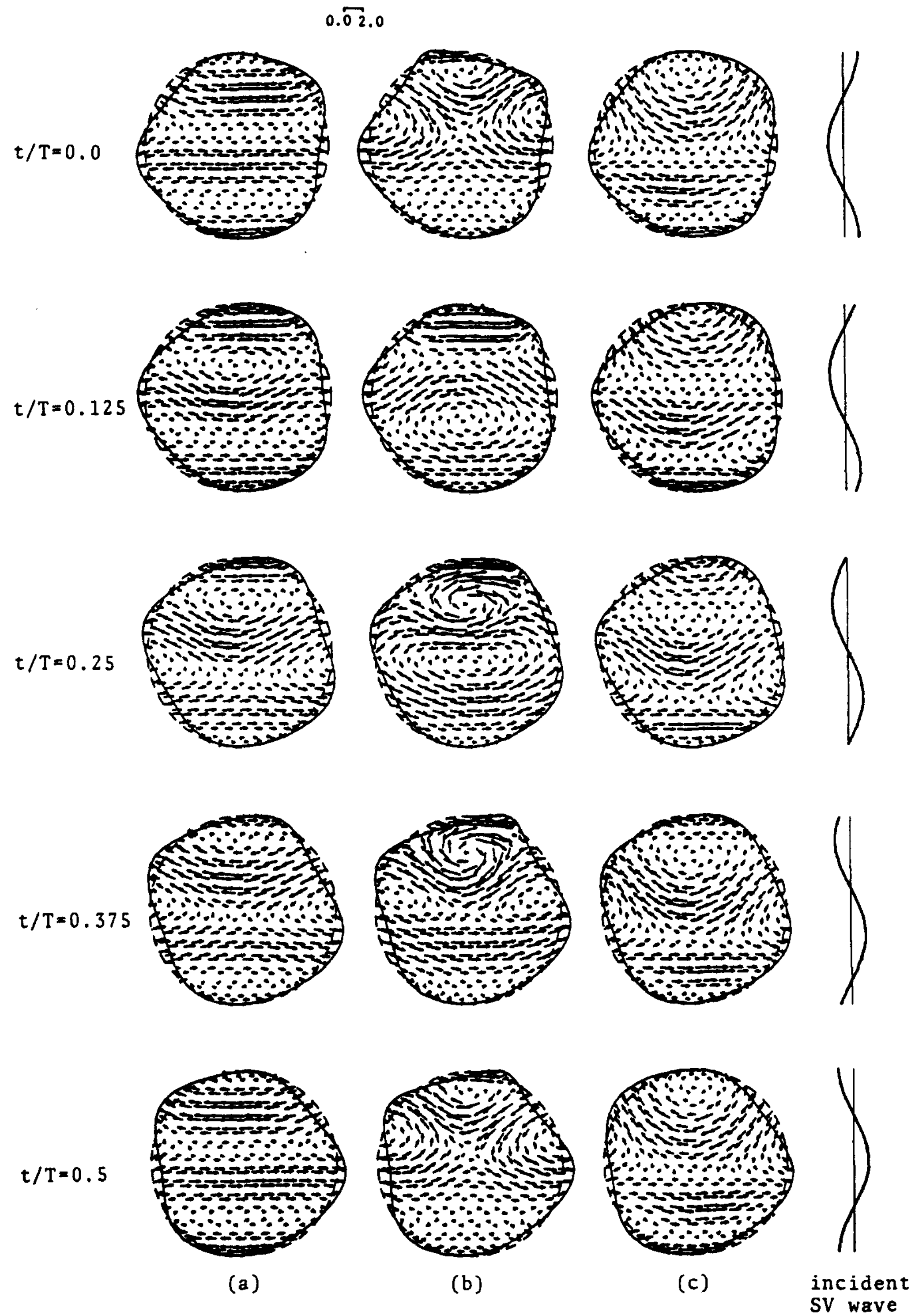

Fig. 5. Deformation of homogeneous (a) and inhomogeneous [(b) and (c)] inclusions subjected to vertically incident SV wave. Figures (a)–(c) correspond to cases 1–3 in the text, respectively.

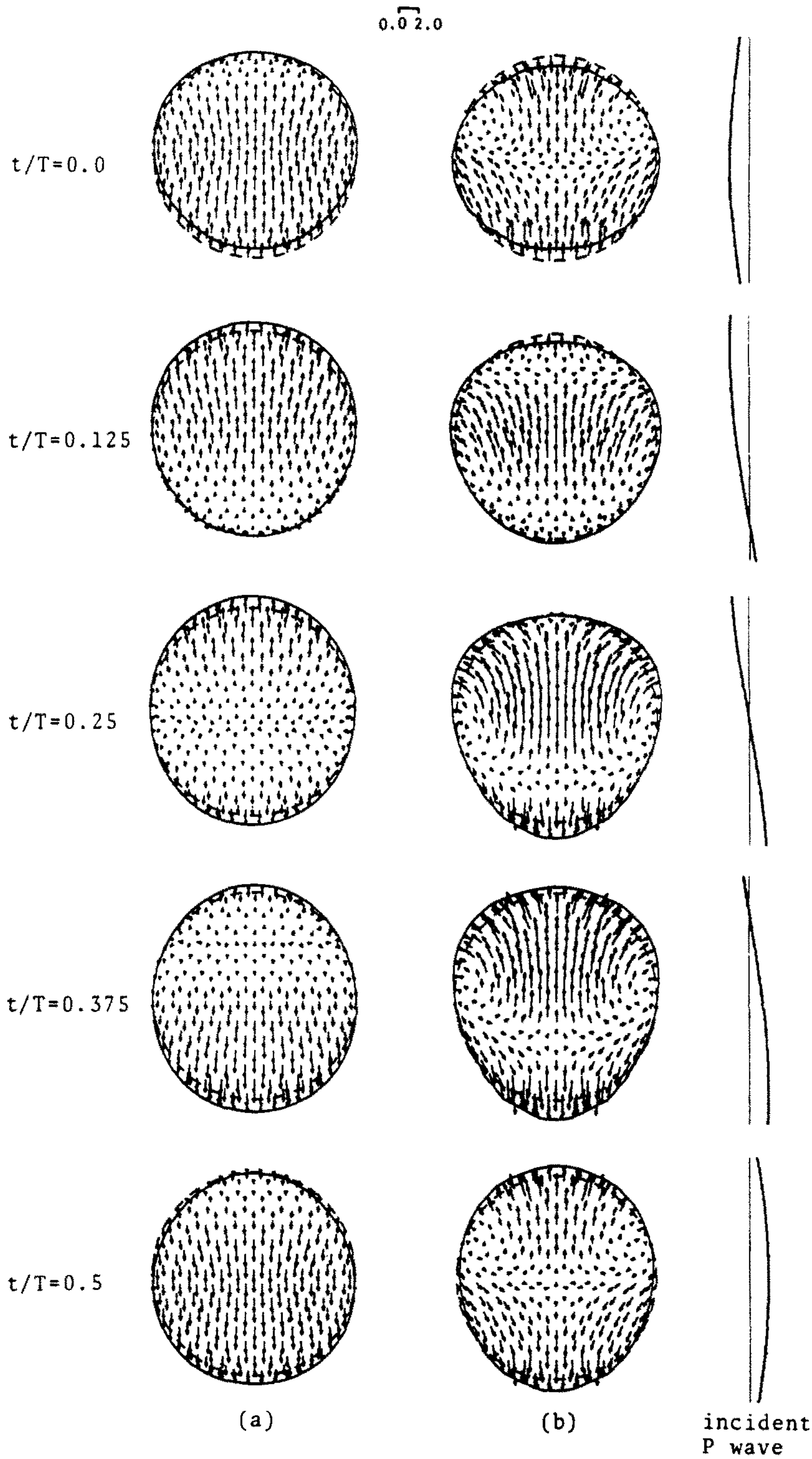

Fig. 6. Deformation of isotropic (a) and anisotropic [(b)–(d)] inclusions subjected to vertically incident P wave. Figures (a)–(d) correspond to cases 4–7 in the text, respectively.

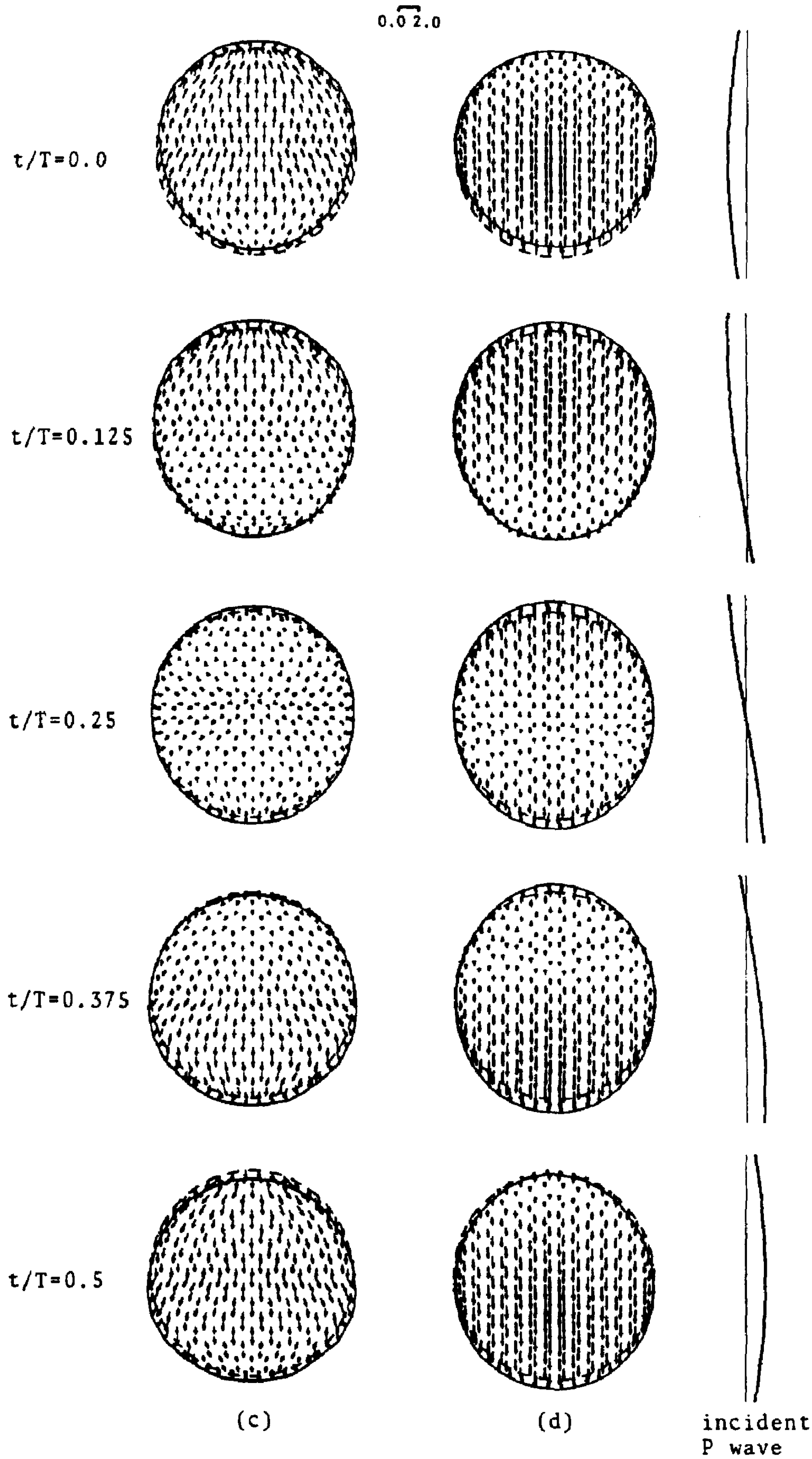

Fig. 6—*continued*

wave. From these figures, we find that the inhomogeneous inclusion shows more complicated features than the homogeneous one. In the upper part of the inclusion in Fig. 5(b), particularly, large amplifications are induced due to the soft properties of material.

5.4.3. *Anisotropic inclusions*. Here, we consider the following four types of homogeneous but anisotropic inclusions.

(case 4)

$$C_{11}/\mu^e = 1.5,\quad C_{12}/\mu^e = 0.5,\quad C_{22}/\mu^e = 1.5,\quad C_{44}/\mu^e = 0.5,\quad \rho^i/\rho^e = 1.0$$

(case 5)

$$C_{11}/\mu^e = 1.5,\quad C_{12}/\mu^e = 0.5,\quad C_{22}/\mu^e = 0.5,\quad C_{44}/\mu^e = 0.5,\quad \rho^i/\rho^e = 1.0$$

(case 6)

$$C_{11}/\mu^e = 1.5,\quad C_{12}/\mu^e = 0.5,\quad C_{22}/\mu^e = 4.5,\quad C_{44}/\mu^e = 0.5,\quad \rho^i/\rho^e = 1.0$$

(case 7)

$$C_{11}/\mu^e = 4.5,\quad C_{12}/\mu^e = 0.5,\quad C_{22}/\mu^e = 1.5,\quad C_{44}/\mu^e = 0.5,\quad \rho^i/\rho^e = 1.0$$

where C_{ij} are Voigt constants[19]. In all these cases, furthermore, Poisson's ratio ν^e in D^e is 0.25 and all other components of C_{ij} are zero. Note that case 4 is identical to the homogeneous isotropic inclusion of case 1. In the following, only inplane motions are taken into account, since inplane motions and antiplane motions are uncoupled in these cases.

Figures 6(a)–(d) show the deformation of inclusions in cases 4–7, respectively. The incident wave is assumed to be a harmonic plane P wave with the wave number $ak_L^e = \pi/(2\sqrt{3})$. A comparison between Figs 6(a) and (b) shows that the phase velocity in the latter inclusion is much slower than the phase velocity in the former one. From Figs 6(a) and (c), on the other hand, we find that the phase velocity in the latter is much faster than that in the former. These phenomena occur due to the differences of the coefficient C_{22} which mainly relates to the longitudinal wave propagating in the x_2-direction. Furthermore, there is little difference between dynamic behaviors of inclusions shown in Figs 6(a) and (d). This shows that the elastic coefficient C_{11} perpendicular to the propagation direction has small effect on the propagation of the P wave.

6. CONCLUDING REMARKS

In the present paper, we have shown the integral formulation of elastodynamic problems for inhomogeneous anisotropic bodies on the basis of a homogeneous fundamental solution in elastostatics. Instead of the static fundamental solution, we can use the dynamic fundamental solution in the integral formulation. In this case, however, there is no advantage over the formulation presented here[1]. One reason is that the dynamic fundamental solution is more troublesome than the static one to deal with it. The other is that the dynamic fundamental solution is inadequate to the dynamic analysis for many frequencies because it involves the frequency parameter implicitly.

Since the governing equation (1) is linear with respect to time, the principle of superposition holds true. Therefore, if the transient wave fields are required, they can be easily reconstructed by superposing steady state solutions by means of the inverse Fourier transform[11]. Furthermore, letting the frequency ω be zero in eqns (26), (29) and (30), we can also treat static problems for inhomogeneous anisotropic bodies.

REFERENCES

1. Y. Niwa and S. Hirose, Scattering of antiplane waves by inhomogeneous alluvial valleys. *Proc. Japan Soc. Civil Engrs, Struct. Engng Earthq. Engng* **2**, 435s–444s (1985).
2. F. J. Rizzo and D. J. Shippy, A method for stress determination in plane anisotropic elastic bodies. *J. Composite Materials* **4**, 36–55 (1970).
3. S. M. Vogel and F. J. Rizzo, An integral equation formulation of three dimensional anisotropic elastostatic boundary value problems. *J. Elasticity* **3**, 203–216 (1973).

4. R. B. Wilson and T. A. Cruse, Efficient implementation of anisotropic three dimensional boundary-integral equation stress analysis. *Int. J. Num. Meth. Engng* **12**, 1383–1397 (1978).
5. N. Nishimura and S. Kobayashi, A boundary integral equation formulation for three dimensional anisotropic elastostatics, *Boundary Elements* (*Proc. 5th Int. Conf.*), pp. 345–354. Springer-Verlag, Berlin (1983).
6. C. C. Lautenbacher, Gravity wave refraction by islands. *J. Fluid Mech.* **41**, 655–672 (1970).
7. F. Mattioli, Wave-induced oscillations in harbours of variable depth. *Computers Fluids* **6**, 161–172 (1978).
8. Y. Niwa, M. Kitahara and M. Yamazaki, Resonance analysis of an inhomogeneous protrusion on a semi-infinite stratum. *Theor. Appl. Mech.* **31**, 245–263 (1982).
9. Y. Niwa, M. Kitahara and S. Hirose, Elastodynamic problems for inhomogeneous bodies, *Boundary Elements* (*Proc. 5th Int. Conf.*), pp. 751–763. Springer-Verlag, Berlin (1983).
10. M. Kitahara, Y. Niwa, S. Hirose and M. Yamazaki, Coupling of numerical Green's matrix and boundary integral equations for the elastodynamic analysis of inhomogeneous bodies on an elastic half-space. *Appl. Math. Modelling* **8**, 397–407 (1984).
11. Y. Niwa, M. Kitahara and S. Hirose, Transient analysis of inhomogeneous structures by integral equation method, *Boundary Elements VI* (*Proc. 6th Int. Conf.*), pp. 6.15–6.26. Springer-Verlag, Berlin (1984).
12. J. Altenbach, Calculation of plates using influence functions for simpler adjoint problems (in German). *ZAMM* **52**, T288–T292 (1972).
13. S. Nemat-Nasser, On elastic, workhardening solids. *Int. J. Solids Structures* **4**, 787–797 (1968).
14. N. Nishimura and S. Kobayashi, Elastoplastic analysis by indirect methods, *Developments in Boundary Element Methods*—3, pp. 59–85. Elsevier Applied Science (1984).
15. H. D. Bui, Some remarks about the formulation of three-dimensional thermoelastoplastic problems by integral equations. *Int. J. Solids Structures* **14**, 935–939 (1978).
16. P. K. Banerjee and R. Butterfield, *Boundary Element Methods in Engineering Science*. McGraw-Hill, London (1981).
17. Y. Niwa, M. Kitahara and H. Ikeda, The BIE approach to transient wave propagation problems around elastic inclusions. *Theor. Appl. Mech.* **32**, 183–198 (1984).
18. Y. Niwa, S. Hirose and M. Kitahara, Application of boundary integral equation (BIE) method to transient response analysis of inclusions in a half space. *Wave Motion* **8**, 77–91 (1986).
19. I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, 2nd edn. McGraw-Hill (1956).